Squaring the Plane

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This research was inspired by two lovely pieces of mathematics. The first is the discovery by William T. Tutte, A. H. Stone, R. L. Brooks, and C. A. B. Smith of squares with integral sides that can be tiled by smaller squares with integral sides, no two alike. Tutte tells the story in “Squaring the Square,” [8], a

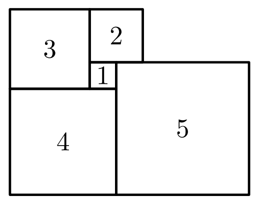

Figure 1: Tutte’s 175×175 perfect square

beautifully written article that conveys vividly the excitement of mathematical research. It became widely-read in 1958 when it was reprinted in Martin Gardner’s “Mathematical Games” column in *Scientific American*. It undoubtedly played a role in inspiring many to become mathematicians.

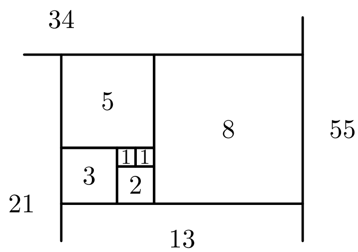

Figure 2: Fibonacci tiling

The second piece is the well-known tiling of the plane by squares whose sides are the Fibonacci numbers (Figure 2).[1]

The tiling is elegant, but in light of Tutte’s work, possesses a minor flaw—it contains two squares of the same size. The flaw is easily remedied. We can use

[1] Coincidentally, this tiling appears in the same volume of Gardner’s columns as “Squaring the Square” [1].

the squared square in Figure 1, together with a square of side 175 (Figure 3).

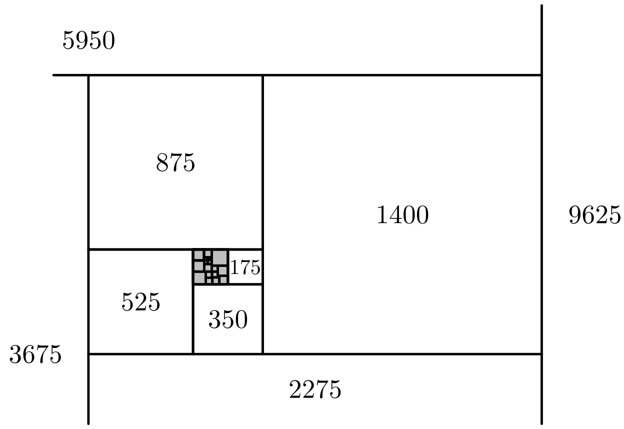

Figure 3: Fibonacci rectangle seeded with Tutte's perfect square

This works, but some of the elegance is lost.

In looking for a natural tiling that doesn't repeat squares, one quickly discovers that the first five squares fit together neatly (Figure 4) but it becomes

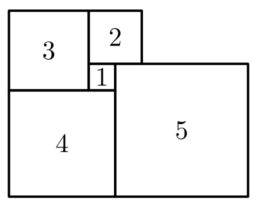

Figure 4: The first five squares

progressively more difficult to add consecutive squares without overlapping or leaving gaps.

This suggests the following question: Can the squares with whole-number sides, one of each size, be fitted together to tile the plane? The answer is that they can.

Theorem *It is possible to tile the plane with non-overlapping squares using exactly one square of each integral dimension.*

In succeeding sections we will prove this theorem, discuss the history of the problem, and pose a number of questions.

1 The Proof

Our proof focuses on rectangles and ells. By "ell," we mean any six-sided figure whose sides meet in right angles.

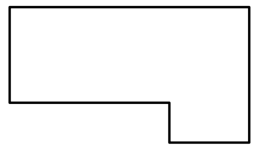

Figure 5: An ell

The following extends Tutte's use of "perfect" for rectangles and squares.

Definition 1 *A figure is* **perfect** *if it is composed entirely of squares of different sizes.*

All the remaining figures in this paper will be perfect.

The key to our proof is Lemma 5, which states that given any perfect ell, it

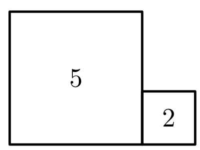

Figure 6: A perfect ell

is possible to add squares to it to form a perfect rectangle.

When we add squares to a figure we'll say we are "puffing it up" if the squares do not appear in the figure. When we puff an ell up to form a perfect rectangle, we'll say we are "squaring up the ell."

We can then "square the plane" as follows:

1. Start with any perfect ell and square it up.
2. Create a new ell by attaching to the rectangle the smallest square not yet used.
3. Square this ell up, making sure that new squares are added in all four directions.
4. Repeat steps 2 and 3 *ad infinitum.*

Definition 2 *An ell in* **standard position** *is an ell oriented so that the single reflex angle is at the upper right. We will refer to the sides by the uppercase letters in Figure 8 and their lengths by the corresponding lower case letters.*

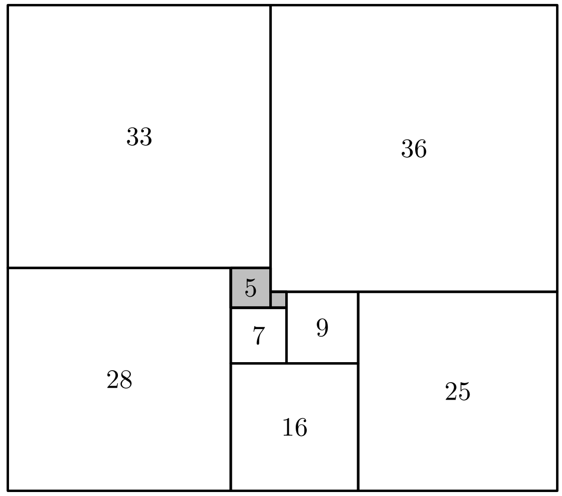

Figure 7: The ell of Figure 6 squared up

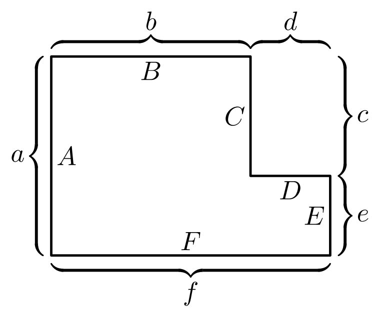

Figure 8: An ell in standard position

When we add squares to a figure, we will always add squares whose side-lengths match that of at least one side of the figure against which it is placed. That means that the only squares we will add to an ell in standard position will be squares adjacent to sides A, B, C, D, E, or F of the appropriate length. We will call these operations **A**, **B**, **C**, **D**, **E**, and **F**. So, for example, applying **B** then **A** to an ell produces a larger ell (Figure 9).

We will denote sequences of operations by sequences of letters with the understanding that they are applied from left to right (so the combination of operations above would be written **BA**).

Note that in any ell in standard position, the length of side A is the sum of the lengths of sides C and E,

$$a = c + e,$$

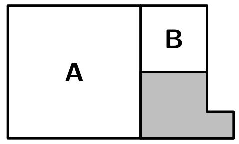

Figure 9: An application of **BA** to the ell of Fig. 8

and the length of side F is the sum of the lengths of sides B and D,

$$f = b + d.$$

Thus, to describe the dimension of such an ell, we need only describe the lengths of sides B, C, D, and E.

Definition 3 *We will say that an ell has* **dimension** $\langle b, c, d, e\rangle$ *if the sides B, C, D, and E of the ell are respectively b, c, d, and e.*

Definition 4 *A side of an ell is* **composite** *if it is not the side of a single square in the ell.*

In squaring up an ell, we will make particular use of the operations **B**, **F**, and **ED**. For that reason, the following definition is most useful.

Definition 5 *A perfect ell is* **regular** *if each of the moves* **B***,* **F***, and* **ED** *either results in a perfect ell in standard position or results in a perfect rectangle.*

If an ell with dimension $\langle b, c, d, e\rangle$ is regular, then there can be no squares with sides b, f, e, or $d + e$ in the ell. In particular, sides B, F, and E must be composite. In addition, it must be that $c \geq d + e$ (or else move **ED** would create an ell which is not in standard position). These properties—no squares of sides b, f, e, or $d + e$, and $c \geq d + e$—are both necessary and sufficient for regularity.

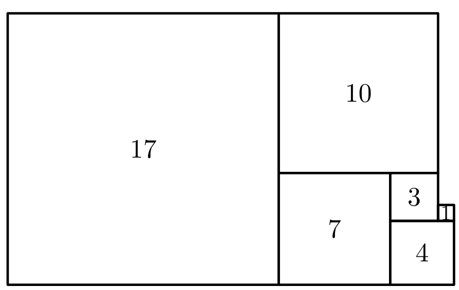

Figure 10: The smallest regular ell has dimension $\langle 27, 12, 1, 5\rangle$

Lemma 1 *Every perfect ell in standard position can be puffed up to form a regular ell without increasing the length of side D.*

Proof: Let the dimension of our ell be $\langle b, c, d, e \rangle$.

We may assume that $a \leq f$, since if not, then side A is the longest side, so **A** can be performed, and in the new ell the length of side A will be less than

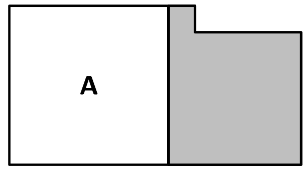

Figure 11: The operation **A**

the length of side F.

Since $a \leq f$, side F is the longest side and **F** can be performed. This allows us to assume further that $c > d$, since if not, we can perform **FABA**, and in the

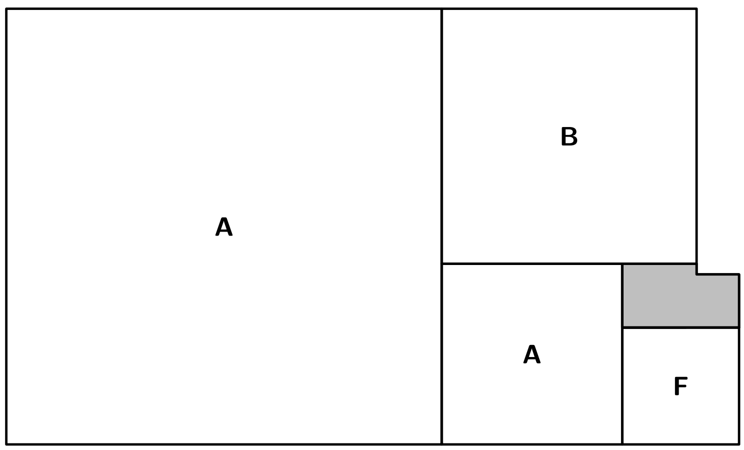

Figure 12: The operation **FABA**

new ell, the length of side C is greater than the length of side D. Note that **FABA** does indeed puff up the ell—the first square, added by **F**, is larger than any in the ell, and each new square is larger than the one before. Note also that the length of side D is unaffected.

With our assumptions now ($f \geq a$, $c > d$), we perform **FABA** (Figure 13).

Let $\langle b', c', d, e' \rangle$ be the dimension of the ell formed after **FABA** (d, since **FABA** doesn't affect side D). It should be clear that performing either **F** or **B** to this new ell would result in a perfect ell in standard position.

To check move **ED**, note first that move **E** adds a square of side $e' = b+d+e$. This is larger than $b + d$ but smaller than $b + c + d + e$, hence a square not previously used. Applying **D** after **E** adds a square of side $d + e' = b + 2d + e$. This will be larger than the square just added, but again less than $b+c+d+e$, since $c > d$. The result then, will be a perfect ell. To show that the ell is either

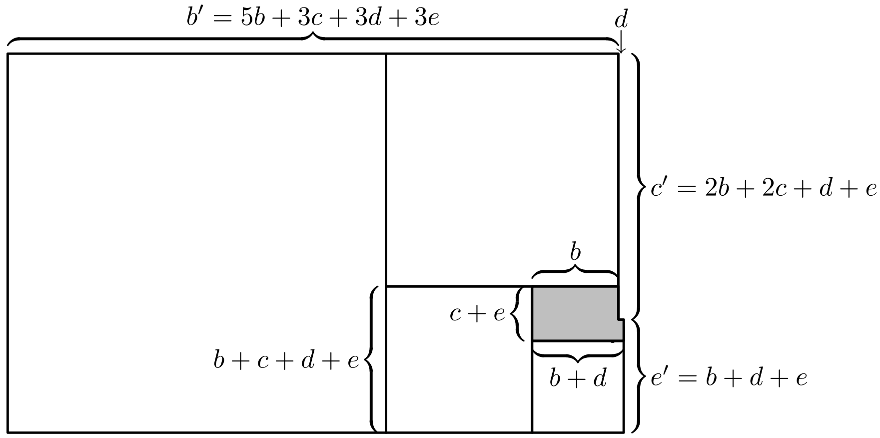

Figure 13: New dimension after **FABA**

a rectangle or in standard position, we need only that $c' \geq d + e'$. But $c' = 2b+2c+d+e$ and $e' = b+d+e$, and using $c > d$, we have $c' = b+2c+e' > d+e'$.

■

Key to proving our main lemma (Lemma 5 on p. 9) is an analysis of $c - e$ mod d. For a regular ell, $c \geq d + e$ so we can write $c = kd + e + i$ where $k \geq 1$ and $i < d$.

Lemma 2 *Suppose we are given a regular ell with dimension $\langle b, c, d, e\rangle$, with $c = kd + e + i$, where $k \geq 2$ and $i < d$. Then the sequence of moves* **BFA** *produces a regular ell with dimension $\langle b', c', d, e'\rangle$ where $c' = (k-1)d + e' + i$.*

Proof: Performing **BFA** (Figure 14) produces an ell with dimension

$$\langle b', c', d, e'\rangle = \langle 3b + c + d + e, b + c, d, b + d + e\rangle.$$

Computing, we have:

$$c' = b + c = b + kd + e + i = (k-1)d + b + d + e + i = (k-1)d + e' + i.$$

To see that the new ell is regular, first note that a move of **B** or **F** would result in a perfect ell in standard position.

To see that **ED** would add only new squares, observe that we have added squares of size $f = b + d$, b, and $a' = 2b + c + d + e$. Move **ED** would add squares of size $e' = b + d + e$ and $d + e' = b + 2d + e$, but we have:

$$2b + c + d + e > b + 2d + e > b + d + e > b + d > b,$$

the first inequality following from $c > d$.

All that remains to show that the ell is regular is that $c' \geq d + e'$, but this follows from $c' = (k-1)d + e' + i$, $k \geq 2$.

■

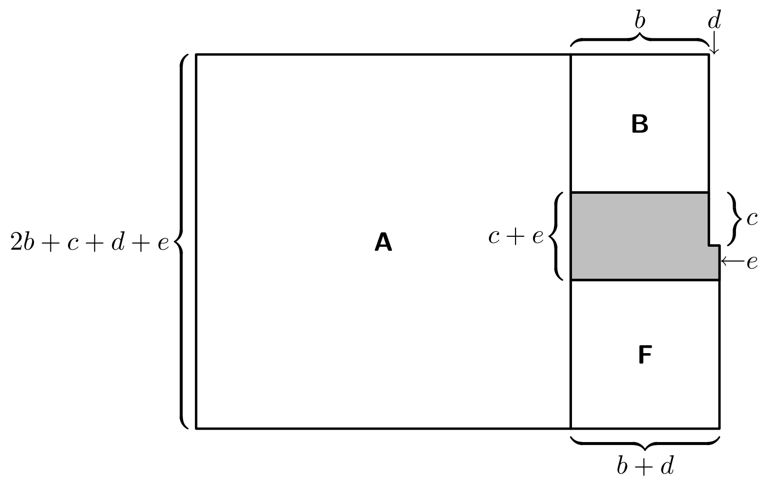

Figure 14: The operation **BFA**

Lemma 3 *If a regular ell with dimension $\langle b, c, d, e\rangle$ is such that $c = kd + e$ for some positive integer k, then the ell can be squared up.*

Proof: We can apply Lemma 2 until we have a regular ell $\langle b', c', d, e'\rangle$ with $c' = d + e'$. This ell can then be squared up with **ED**.

■

Lemma 4 *Every regular ell can be squared up.*

Proof: Given a regular ell with dimension $\langle b, c, d, e\rangle$, we have $c = kd + e + i$, for some $k \geq 1$ and $i < d$. We will show that the ell can be squared up by general induction on d.

If $d = 1$, then $i = 0$ and we are done by Lemma 3.

Now assume that the result holds for all $d^* < d$. Apply Lemma 2 iteratively until we have an ell of dimension $\langle b', c', d, e'\rangle$ with $c' = d + e' + i$. Then **ED** produces an ell with dimension $\langle b', i, d + e', d + 2e'\rangle$ (see Figure 15).

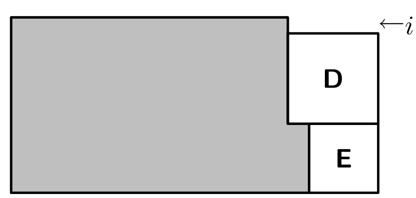

Figure 15: The operation **DE**

By flipping this over we obtain in standard position an ell with dimension $\langle 2e' + d, e' + d, i, b'\rangle$ (see Figure 16).

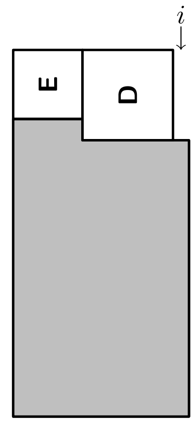

Figure 16: A flip after **DE**

By Lemma 1 we can puff this up to a regular ell without increasing the length of side D (which is i). Since $i < d$, by the inductive hypothesis this new ell can be squared up.

■

Our main lemma follows immediately:

Lemma 5 *Every perfect ell can be squared up.*

Proof: Given an ell, we puff it up to a regular ell by Lemma 1, then square it up by Lemma 5.

■

In other words, every perfect ell is a part of a perfect rectangle.

Proof of the Theorem: It is easy to see how to carry out the plan described earlier. We start with any perfect ell and grow it by adding squares as described in the algorithm on p. 3.

Note that we don't move squares around; once added to the figure they are fixed. In particular, we never actually flip the figure as in the proof of Lemma 5—the flipping described there is simply to show that we can square up a particular ell.

Note also that we are guaranteed to fill the plane since we are careful to grow the figure in all four directions in each cycle of the algorithm.

And finally, infinity being what it is, we are guaranteed to incorporate in the tiling a square of each integral dimension.

■

2 Reflections

In preparing this paper, we learned some of the history of the problem. The question was first posed by Solomon Golomb in a 1975 article in the *Journal* *of Recreational Mathematics*. He called it the "heterogeneous tiling conjecture" and challenged readers to prove or disprove it.

Martin Gardner wrote about the question four years later in his column in *Scientific American* [2]. He described one approach to a solution, a fairly orderly tiling of roughly three-quarters of the plane with squares and reported that Verner Hoggatt Jr, then editor of *The Fibonacci Quarterly* had showed that no square in the tiling appeared more than once.

Grünbaum and Shepard wrote about the problem in their 1987 book, *Tilings and Patterns* [4]. They described there a second way in which a squared square S can addition generate a tiling of the plane (in addition to the method shown on p. 2): Take a second copy of S and expand it to a square S_1 such that the smallest square in S_1 is the size of the original square S and fit S into that square. Take another copy of S and expand it to S_2 so that its smallest square is the size of S_1, and so on.

Grünbaum and Shepard record the observation of Carl Pomerance that in every tiling of the plane by unique squares known at that time, the sizes of the squares grow exponentially.

In 1997, Karl Scherer [5] succeeded in tiling the plane using multiple copies of squares of all integral sizes. The number, $t(n)$ of squares of size n is finite but not bounded. He describes his tiling as "size-alternating," in that no two squares of the same size share any portion of an edge (though they may share a corner).

3 Questions

Tiling is an enormous field. This theorem might be said to reside in the subfield that concerns infinite tilings of the plane that use exactly one specimen each of a well-defined collection of similar figures. Much work has been done here and there is much to do. Here are a few directions for research:

Efficiency: The algorithm presented in this paper is extravagant in that the ratio of the largest square used so far to the smallest square not yet used rapidly diverges. The procedure for squaring up, for example, when applied to the smallest possible ell, a 2×2 square next to a 1×1 square, ends in a rectangle of size:

$$1106481365205154721693 \times 2659648557852203795117$$

The smallest square not used at this point is 4×4.

The squaring-up procedure can certainly be improved. By way of illustration, take the ell in Figure 6. This can be squared up to a 69×61 rectangle with the sequence of operations **FEFEABC** (see Figure 7). Our procedure, however, doesn't square the ell up until the size is approximately $(5.0 \times 10^{14272}) \times (5.8 \times 10^{14272})$.

Can our squaring-up procedure be improved in some well-defined way? Does an algorithm exist for tiling the plane that methodically expands a connected island of squares in such a way that the ratio of of the largest square used to the smallest not yet used is bounded by a polynomial?

Simple tilings: A perfect figure is **simple** if it contains no perfect subrectangle. Our tiling is far from simple. Is there a simple tiling of the plane using one specimen of each integral size?

The half-plane and quarter plane: Can the half-plane be squared? Can the quarter-plane be squared? We are especially interested in this question because if it is possible to tile a quarter plane four times using, altogether, every integral square just once, then it's possible to tile the plane using all the integral squares plus one square of any given size (say $\pi \times \pi$).

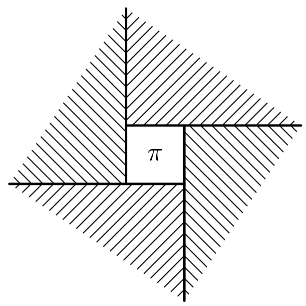

Figure 17: $\pi \times \pi$

Rational squares: The algorithm of our theorem can easily be used to tile the plane using one of every rational square. As with integral squares, we don't know if a similar procedure works for the half- or quarter-plane.

Positive and negative squares: We can tile the plane with squares whose sides are natural numbers and with squares whose sides are rationals. What about squares whose sides are (positive and negative) integers? We interpret the effect of placing a small negative square on top of a large positive one as removing a part of the large square. Once again, our algorithm works easily for this. Just as placing a positive square next to a rectangle creates an ell, so does placing a negative square on a corner of a rectangle. With care, no point of the plane will be touching more than 3 squares (one negative, two positive).

Odd squares: Can the plane be squared with all the odd squares? This seems unlikely to us. Can the plane be squared with *some* of the odd squares? In general, what well-defined subsets of the natural numbers can be used to tile the plane?

Coloring: Neither our tiling nor the Fibonacci tiling can be 3-colored. Is there a 3-colorable tiling of the plane using exactly one square of each integral size? Is there a 3-colorable tiling of the plane using no more than one square of each integral size? Is there a simple algorithm for 4-coloring the tiling described in this paper?

Space: Can space be cubed?

Triangles: Scherer proved that the plane cannot be tiled with equilateral triangles of different sizes if one triangle is smallest ([6]). He has found a way of tiling the plane, however, with different sizes of iscoceles right triangles and with enlargements of certain other triangles [7]. Left open is the question: can the plane be tiled with all rational equilateral triangles so that no triangle has an infinite number of neighbors?

References

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